

Exercise Sheet 8

Exercise 1. Let $\pi : (X, x_0) \rightarrow (Y, y_0)$ be a covering map between connected, path-connected, locally path-connected topological spaces. Set

$$G := \pi_1(Y, y_0), \quad H := \pi_* (\pi_1(X, x_0)) \leq G,$$

and write $\text{Deck}(\pi)$ for the group of deck (covering) transformations. Let $F = \pi^{-1}(y_0)$ be the fiber over y_0 . Given a loop $\gamma : [0, 1] \rightarrow Y$ based at y_0 , and a point $x \in F$, there is a unique lift $\tilde{\gamma}_x$ of γ starting at x , i.e. $\tilde{\gamma}_x(0) = x$ and $\pi \circ \tilde{\gamma}_x = \gamma$. The endpoint of the lift $\tilde{\gamma}_x$ lands again in the fiber F . Therefore loops in Y induce permutations in the fiber F . Taking homotopy classes, this defines the monodromy representation

$$\rho : G \rightarrow \text{Sym}(F), \quad \rho([\gamma])(x) := \tilde{\gamma}_x(1) \in F.$$

- (1) Show that $\rho(G)$ acts transitively on the fiber F .
- (2) Show that $H = \text{Stab}_G(x_0) = \{g \in G : \rho(g)(x_0) = x_0\}$.
- (3) Show that the coset space G/H is in bijection with F .
- (4) Show that

$$\ker(\rho) = \bigcap_{x \in F} \text{Stab}_G(x) = \bigcap_{g \in G} gHg^{-1}.$$

- (5) Show that if a deck transformation $\tau \in \text{Deck}(\pi)$ fixes some $x \in X$, then $\tau = \text{id}_X$. Show that every $\tau \in \text{Deck}(\pi)$ induces a bijection $\tau|_F : F \rightarrow F$ and conclude that

$$\Phi : \text{Deck}(\pi) \rightarrow \text{Sym}(F), \quad \Phi(\tau) = \tau|_F,$$

is injective.

- (6) Show that $\Phi(\text{Deck}(\pi))$ is the centralizer of monodromy, that is,

$$\begin{aligned} \Phi(\text{Deck}(\pi)) &= \text{Cent}_{\text{Sym}(F)}(\rho(G)) \\ &= \{\sigma \in \text{Sym}(F) : \sigma\rho(g) = \rho(g)\sigma \text{ for all } g \in G\}. \end{aligned}$$

Exercise 2. Let F be a set and let $A \leq \text{Sym}(F)$ act transitively on F . Let $C = \text{Cent}_{\text{Sym}(F)}(A)$ be the centralizer of A in $\text{Sym}(F)$. Show that

- (1) C acts freely on F .
- (2) C acts transitive on F if and only if A acts freely on F .

Exercise 3. Let $\pi : (X, x_0) \rightarrow (Y, y_0)$ be a covering map between connected, path-connected, locally path-connected topological spaces. Set

$$G := \pi_1(Y, y_0), \quad H := \pi_* (\pi_1(X, x_0)) \leq G,$$

and write $\text{Deck}(\pi)$ for the group of deck (covering) transformations. Let $d = |\pi^{-1}(y_0)| = \text{deg}(\pi)$ and let $\rho : G \rightarrow \text{Sym}(\pi^{-1}(y_0)) \cong S_d$ be the monodromy representation on the fiber $F = \pi^{-1}(y_0)$. Prove that the following are equivalent (any may be taken as the definition of a Galois (regular/normal) covering):

- (1) $\text{Deck}(\pi)$ acts transitively on the fiber F .
- (2) H is a normal subgroup of G .
- (3) There exists a free, proper action of a discrete group Γ on X with $Y \cong X/\Gamma$ and π the quotient map.
- (4) The monodromy image $\rho(G) \subset S_d$ acts freely and transitively on the fiber F .

When these equivalent conditions hold, we have group isomorphisms

$$\text{Deck}(\pi) \cong \Gamma \cong G/H \cong \rho(G),$$

and $|\text{Deck}(\pi)| = \text{deg}(\pi)$.

Exercise 4. Let $f : X \rightarrow Y$ be a nonconstant holomorphic map between connected Riemann surfaces. Let $B \subset Y$ be the set of branch values and put $U := Y \setminus B$ and $X^\circ := f^{-1}(U)$. We say that f is a branched Galois (regular/normal) covering if the unbranched covering

$$f|_{X^\circ} : X^\circ \rightarrow U$$

is Galois (regular/normal). Let τ be a deck transformation of the unbranched covering $f|_{X^\circ} : X^\circ \rightarrow U$.

- (1) Show that there exists a unique biholomorphism $\bar{\tau} \in \text{Aut}(X)$ such that

$$\bar{\tau}|_{X^\circ} = \tau \quad \text{and} \quad f \circ \bar{\tau} = f \quad \text{on } X.$$

Deduce that the deck group on X° and the extended deck group on X are isomorphic.

- (2) Show that for every $x \in X$,

$$e_{\bar{\tau}(x)}(f) = e_x(f),$$

i.e. the ramification multiplicity is invariant under deck transformations.

- (3) Now assume that f is Galois. Show that for each $b \in B$, the (extended) deck group $\text{Deck}(f)$ acts transitively on the branched fiber $f^{-1}(b)$. Deduce that for each $b \in B$, all points in $f^{-1}(b)$ have the same ramification multiplicity.

Exercise 5. (for credit, due on 16 November) (5 points)

Fix $n \in \mathbb{Z}_{\geq 1}$ and consider the map $f_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by

$$f_n(z) = \frac{z^{2n} + 1}{z^n}.$$

Show that f_n is a branched Galois covering whose group of deck transformations is isomorphic to the dihedral group:

$$\text{Deck}(f_n) \cong D_{2n} = \langle \sigma, \rho : \rho^n = \sigma^2 = 1, \sigma\rho\sigma = \rho^{-1} \rangle.$$

Exercise 6. Find a branched covering $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ that is not Galois.